Structure functions in the stochastic Burgers equation

F. Hayot and C. Jayaprakash
Department of Physics, The Ohio State University, Columbus, Ohio 43210
(Received 20 December 1996)

We study analytically and numerically structure functions $S_q(r)$ in the one-dimensional Burgers equation, driven by noise with variance $\propto|k|^\beta$ in Fourier space, (a) when the noise is cut off at some length l_c, and (b) when it is not. We present exact relations satisfied by $S_3(r)$ (the von Karman–Howarth relation) and $S_4(r)$ that form the basis of our analysis. When there is a cutoff length, shocks occur and $S_q(r)\propto r$ for $q\geqslant 2$ for $\delta<r<l_c$ where δ is the shock thickness for all β between -1 and 2. We deduce this behavior from the exact relations along with an ansatz that is verified numerically. When there is no cutoff length, multifractal behavior is known to occur only when $\beta<0$. Through a study of exact expression S_3 we highlight the difference between multifractality in this case as compared to the case with a cutoff. [S1063-651X(97)02807-9]

PACS number(s): 05.45.+b

Scaling and multifractality in extended dynamical systems is an important area of study. An issue of interest is the mechanism for such behavior in driven, deterministic versus stochastic systems. The possibility of describing deterministic partial differential equations (PDEs) by effective stochastic equations on appropriate length scales raises important related questions. The exploration of these questions is obviously easier in one spatial dimension. The stochastic Burgers [1] equation is a simple stochastic PDE that has attracted wide attention recently to explore various of these issues [2–5]. The one-dimensional equation reads

$$\partial u/\partial t+u\partial u/\partial x=\nu\nabla^2u+\eta, \tag{1}$$

where $u(x,t)$ is the velocity field, ν the viscosity, and $\eta(x,t)$ a Gaussian noise, of zero mean, with a variance in Fourier space given by

$$\langle\hat{\eta}(k)\hat{\eta}(k')\rangle=\hat{D}(k)\delta_{k+k',0}\delta(t-t'), \tag{2}$$

where $\hat{D}(k)=2D_0|k|^\beta$. For the determination of multifractality, the principal object of study is the structure function

$$S_q(r)=\langle[u(x+r,t)-u(x,t)]^q\rangle\sim r^{\zeta_q}. \tag{3}$$

For the stochastic Burgers equation, the connection between multifractality—the fact that the ζ_q do not grow linearly with q—and the presence of shocks in the velocity field has been investigated earlier: We know from our previous studies [6] that, when β in Eq. (2) is positive $(0<\beta<2)$ multifractal behavior does not occur and velocity profiles from numerical simulations show no evidence for shocks, whereas both shocks and multifractality occur for β negative $(-1\leqslant\beta<0)$. The result at $\beta=-1$ has been obtained first by Chekhlov and Yakhot [2]. We also know from the work of Bouchaud, Mézard, and Parisi [4] that (for high space dimension) when $\beta=2$ and there is, moreover, a cutoff length l_c, multifractality occurs for distances much smaller than l_c. Analytic work based on field-theoretic methods has been pursued in Ref. [5].

We present here the results of an analytic and numerical study of multifractal behavior for noises of the form (2). We give an extensive discussion based on the analog of the von Karman–Howarth relation [7] for $S_3(r)$, and the next relation in an infinite hierarchy of such equations for the structure functions. These illuminate the origin and limits of validity of multifractality in the stochastic Burgers equation for noise variances with or without a cutoff.

We assume that asymptotically the system reaches a (temporally) steady state that is spatially homogeneous upon averaging over noise. In this case the von Karman–Howarth relation [7] can be derived easily from the equation of motion; the deterministic case is discussed, for example, in Ref. [8]. We obtain

$$\frac{dS_3(r)}{dr}=-6\langle u(x)[\eta(x+r)+\eta(x-r)]\rangle -12\nu\frac{d^2}{dr^2}\langle u(x)u(x+r)\rangle. \tag{4}$$

We can use the results of Novikov and Donsker [9] for a Gaussian random ensemble that imply $\langle\hat{u}(k)\hat{\eta}(-k)\rangle=\frac{1}{2}\hat{D}(k)$ and rewrite the above relation in the following useful form:

$$\frac{1}{6}\frac{dS_3(r)}{dr}=\nu\frac{d^2S_2(r)}{dr^2}-\frac{1}{L^2}\sum_k\hat{D}(k)\cos kr. \tag{5}$$

Observe that the sum on k of the noise variance can depend on the ultraviolet cutoff when the noise itself occurs at all scales.

We can similarly derive an exact relation (valid for noise with and without cutoff) between the four- and three-point functions, that is, the next one in an infinite hierarchy of such equations. This can be derived by considering the time derivative of $\langle u^2(x_1)u(x_2)-u(x_1)u^2(x_2)\rangle$ where $x_1=x+r$ and $x_2=x$, using the equation of motion and averaging over space and the Gaussian noise ensemble. We exhibit below a particular version that can be recast into other interesting forms by algebraic manipulations:

$$\frac{1}{6}\frac{dS_4(r)}{dr}=\nu\langle[\partial_1^2u(x_1)-\partial_2^2u(x_2)][u(x_1)-u(x_2)]^2\rangle. \tag{6}$$

In the preceding, ∂_1 represents $\partial/\partial x_1$, etc. We draw attention to the fact that the noise term does not contribute to this equation: the derivation leads to terms of the form $\langle \eta(x+r)u^2(x)\rangle$ which can be argued to vanish.

The infinite hierarchy can be obtained by considering the time derivative of the generating function $\langle e^{a(u_1-u_2)}\rangle$ for a real constant a. This leads to

$$\frac{d}{dr}\left\langle\left(u_1-u_2-\frac{2}{a}\right)e^{a(u_1-u_2)}\right\rangle$$
$$=\nu\{a\langle(\partial_1^2u_1-\partial_2^2u_2)e^{a(u_1-u_2)}\rangle$$
$$+a^2\langle(\partial_1u_1)^2+(\partial_2u_2)^2\rangle\langle e^{a(u_1-u_2)}\rangle\}$$
$$-\frac{a^2}{L^2}\sum_k \hat{D}(k)\cos(kr)\langle e^{a(u_1-u_2)}\rangle. \quad (7)$$

We will use these relations and plausible arguments to elucidate the behavior of the structure functions $S_2(r)$, $S_3(r)$, and $S_4(r)$, in the limit of $\nu \to 0$ and comment on higher-order structure functions.

NOISE WITH CUTOFF

We consider the case where there is a cutoff l_c ($l_c<L$) in the noise variance: $\hat{D}(k)=0$ for $|k|>2\pi/l_c$. We assume that as the system size increases, the cutoff scale also increases. We find numerically that multifractality occurs in the inertial range delineated by length scales smaller than l_c, the scale set by the noise cutoff and larger than the shock thickness δ, whatever the value of β between 2 and -1. In particular, $S_q(r)$ varies linearly with r for $q\geqslant 2$. We used a pseudospectral code as described in Ref. [6] to solve the equation numerically. In Fig. 1(a) $S_2(r)$ and $|S_3(r)|$ are displayed and in Fig. 1(b) $S_4(r)$ is plotted as a function of r for $\beta=-\frac{1}{2}$. The numerical results are consistent with linear behavior in the inertial range. We have checked that similar results are obtained for $\beta=\frac{1}{2}$ and $\beta=2$ and larger q. For distances smaller than δ, the structure functions show the expected Gaussian behavior. We provide below a theoretical understanding of the results in the inertial range.

In the limit of $\nu\to 0$, which we consider in the ensuing discussion, the shock thickness δ is of the order of ν/V [8] where V is the characteristic velocity scale, typical of the ''jump'' across the shock. In this case, the first term in Eq. (5) proportional to ν does not contribute in the inertial range. (On the other hand, for $r\to 0$, the two terms cancel each other; see later for more comments on this term.) Thus in the region where shocks dominate the von Karman–Howarth relation yields

$$\frac{dS_3(r)}{dr}=-6\frac{1}{L^2}\sum_{k=k_{\min}}^{k_c}\hat{D}(k)\cos kr\approx -6\frac{1}{L^2}\sum_{k=k_{\min}}^{k_c}\hat{D}(k), \quad (8)$$

where $k_c=2\pi/l_c$ denotes the upper cutoff in k space, $k_{\min}=2\pi/L$. For r in the inertial range $r\ll l_c$, we also have $rk\ll 2\pi$ in the argument of the cosine and the second approximate equality in Eq. (8) follows. Integrating the equation yields

$$S_3(r)=-12\epsilon_0 r/L, \quad (9)$$

where $\epsilon_0\equiv(1/2L)\Sigma_k\hat{D}(k)$ is the total rate of energy input. The proportionality factor in Eq. (9) agrees with the numerical results within about 20%. We can show that the linearity of $S_3(r)$ corresponds to the energy flux in k space being a constant and thus the ''mechanism'' is similar to the case of the Navier-Stokes equation with nonrandom forcing.

Our result in Eq. (9) is in agreement with the result of Polyakov [5] obtained by pointsplitting methods in the limit $\nu\to 0$. We have presented the more general relation [Eq. (5)] and specified the conditions under which the linearity of S_3 obtains.

We consider S_2 and S_4 next. The even-numbered structure functions are even in r and increase nonanalytically as $|r|$. We explore this behavior analytically next.

The equation of motion, Eq. (1), automatically yields energy conservation: the energy supplied by the stochastic noise is dissipated by the action of viscosity.

$$\frac{\nu}{L}\sum_k k^2\langle\hat{u}(k)\hat{u}(-k)\rangle=\frac{1}{2L}\sum_k\hat{D}(k)\propto D_0(1/l_c)^{1+\beta}. \quad (10)$$

Note that energy input is independent of short-distance behavior and depends only on l_c. In the inertial range both viscous dissipation and energy input due to the stochastic driving are negligible. The energy cascades to smaller scales

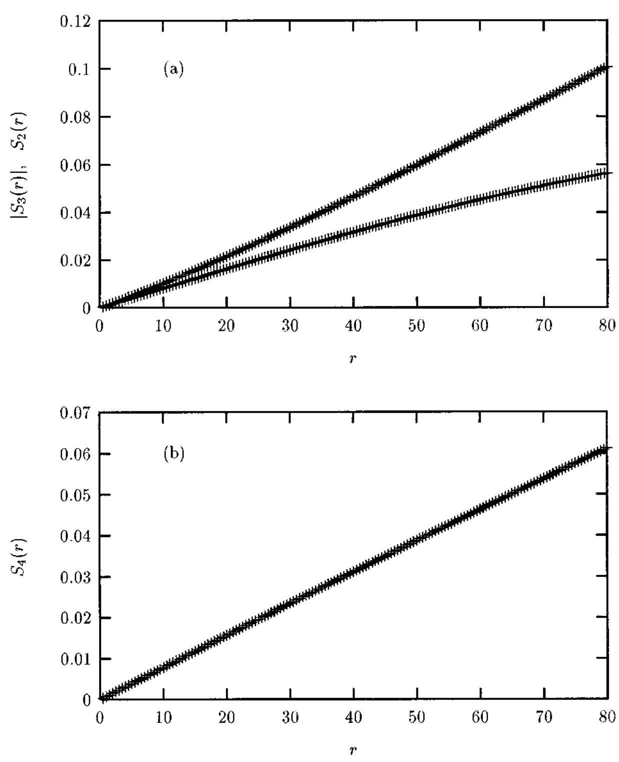

FIG. 1. (a) The structure function $|S_3(r)|$ (lower line) and $S_2(r)$ (upper line) vs r showing linearity in the inertial range. Distances are measured in units in which the system size is $L=1024$. There is a smooth cutoff at $k_c=4(2\pi/L)$. We have used $\beta=-0.5$, $\nu=0.05$, and $D=2.0\times 10^{-6}$. (b) $S_4(r)$ vs r. The linearity in the inertial range is evident.

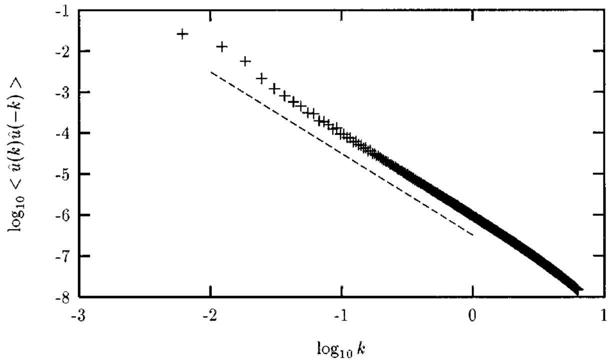

FIG. 2. The energy spectrum $\langle \hat{u}(k)\hat{u}(-k)\rangle$ vs k for the same system as in Fig. 1 on a log-log plot. The wave vector k is in units of the basic interval in k space, $2\pi/L$. The dashed line has a slope of -2. This leads to $S_2(r)\propto|r|$ as discussed in the text and seen in Fig. 1(a).

and we can define a dissipation wave number K_d at which the dissipation defined by $2\nu\int_0^{K_d}dk\,k^2\langle\hat{u}(k)\hat{u}(-k)\rangle$ becomes comparable to the energy flux. Substituting the scaling form $\langle\hat{u}(k)\hat{u}(-k)\rangle\propto|k|^{-\zeta}$ yields the dissipation (up to K_d) proportional to $\nu K_d^{3-\zeta}$; since this is comparable to the energy flux that is independent of K_d, we conclude that $\nu \propto K_d^{\zeta-3}$. If we now make the reasonable identification of the inverse of the dissipation wave number or the dissipation scale with the shock thickness, δ, we obtain $\nu\propto\delta^{3-\zeta}$. Now the shock thickness vanishes in the zero viscosity limit, as mentioned earlier, as ν/V. This allows one to deduce that δ=2 and therefore

$$\langle\hat{u}(k)\hat{u}(-k)\rangle\propto|k|^{-2}.$$

We have checked this expectation numerically; the exponent -2 gives a very good fit to the data for the energy spectrum shown in Fig. 2. We emphasize that we determine the exponent governing the energy spectrum by identifying the dissipation scale with the shock thickness and using the way in which this scale depends on ν for small ν.

The form $\langle\hat{u}(k)\hat{u}(-k)\rangle = AV^2/k^2$ for the energy spectrum obtained above leads directly to linearity of $S_2(r)$ in the inertial range. Fourier transformation yields $S_2(r) = (1/L^2)\Sigma_k\langle\hat{u}(k)\hat{u}(-k)\rangle 2(1-\cos kr)$; in the large L limit, the leading behavior is given by $S_2(r)=AV^2|r|/L$ as seen in Fig. 1(a).

We next examine the behavior of $S_4(r)$ and address how the exact relation Eq. (6) leads to linear behavior in the inertial range. It is convenient to recast the equation into the following form:

$$\frac{1}{6}\,\frac{dS_4(r)}{dr}=\frac{2}{3}\,\nu\,\frac{d^2S_3}{dr^2}-\frac{4i\nu}{L^3}\sum_{k,q}C_3(k,q)q(k+q)\sin(kr). \tag{11}$$

In the preceding we have defined the three-point correlation function in wave-vector space $C_3(k,q) \equiv\langle\hat{u}(k)\hat{u}(q)\hat{u}(-k-q)\rangle$. Note that only the imaginary part of the three-point function contributes to the sum in Eq. (11). In the inertial range, when all the wave vectors are smaller than $2\pi/\delta$ and larger than $2\pi/l_c$ we make the following ansatz:

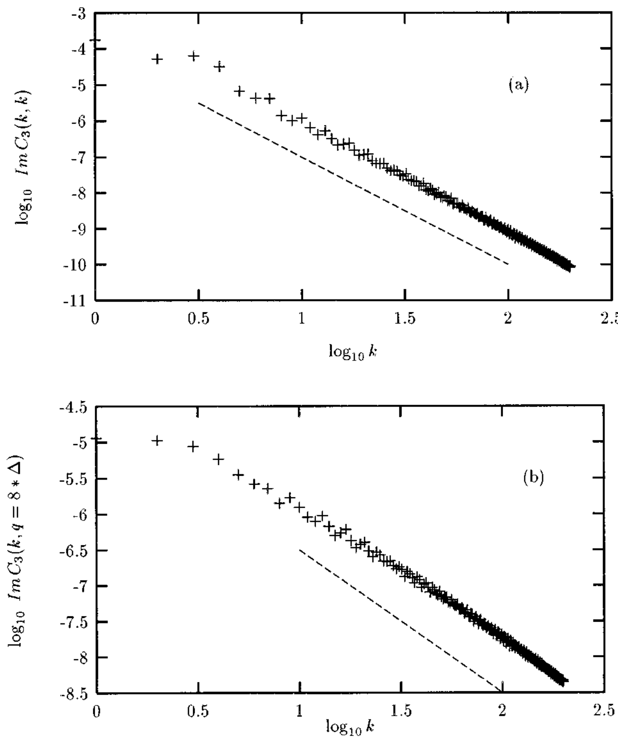

FIG. 3. (a) $\log_{10}\mathrm{Im}C_3(k,k)$ vs $\log_{10}k$. The three-point function is defined in the text. The dashed line corresponds to a slope of -3 as predicted by our ansatz in Eq. (12). (b) $\log_{10}\mathrm{Im}C_3(k,q=4\Delta)$ vs $\log_{10}k$ where $\Delta=2\pi/L$ is the interval in k space. Note that the slope of the dashed line is -2. The slope of the graph deviates from -2 as k becomes comparable to q as predicted by the ansatz.

$$\mathrm{Im}[\langle\hat{u}(k)\hat{u}(q)\hat{u}(-k-q)\rangle]=\frac{A_3V^3}{kq(k+q)}. \tag{12}$$

The structure function S_3 is related to $\langle\hat{u}(k)\hat{u}(q)\hat{u}(-k-q)\rangle$ by Fourier transformation. Using Eq. (12) it is easy to verify that the leading behavior is given by $S_3(r)=-A_3V^3(r/L)$. This confirms that the ansatz correctly reproduces the known result. One can use the ansatz in Eq. (11) and after some algebraic manipulations find a ν-independent contribution: $dS_4/dr\propto c_3V^4\,\mathrm{sgn}(r)/L$. This gives rise to the $|r|$ behavior of S_4. Thus we have provided an analytic understanding of the structure functions for q=2, 3, and 4.

We have studied $C_3(k,q)$ numerically and checked that the results are consistent with our conjectured form; in Fig. 3(a) we show $\mathrm{Im}C_3(k,k)$ which decreases as k^{-3} while in Fig. 3(b) we display $\mathrm{Im}C_3(k,q)$ for a fixed q which for moderate k's decreases as k^{-2} as predicted by Eq. (12).

NOISE WITHOUT CUTOFF

Consider the case in which there is no cutoff, i.e., the noise variance has the functional form $|k|^{\beta}$ for all wave vectors up to the ''ultraviolet'' cutoff. In this case, the system behaves differently according to whether β is positive or negative. When β is positive there is no multifractality, and no region where in the von Karman–Howarth relation the term proportional to viscosity can be neglected. In particular, in the case β=2, the existence of the fluctuation-dissipation theorem leads to $\langle \hat{u}(k)\hat{u}(-k)\rangle = 2D_0/\nu$ and the two terms in Eq. (5) cancel each other and yield $dS_3/dr = 0$. When β is negative, multifractality is known to occur, from numerical simulations, up to distances of the order of system size (limited by the average distance between shocks), as we have shown previously [6]. Next we comment on the behavior of S_2 and study analytically S_3 for negative β and compare them to the case with a cutoff.

In contrast to the cutoff case in the earlier section, we do not know the dependence of the short-distance cutoff on ν in the limit of vanishing viscosity. Therefore energy balance cannot be used to obtain the scaling exponent of the energy spectrum $\langle \hat{u}(k)\hat{u}(-k)\rangle$, which in turn determines $S_2(r)$; in this case as discussed in our earlier work we appeal to renormalization group calculations [10] and extrapolate naively from the β positive ($0<\beta<1.5$) regime to negative β. This yields $\zeta_2 = -2\beta/3$ and we have $S_2(r) \sim r^{-2\beta/3}$. For $\beta = -1$ for example, $S_2(r) \sim r^{2/3}$, as pointed out in Ref. [2] as compared with the $|r|$ behavior when there is a cutoff.

The leading behavior of $S_3(r)$ in the inertial range for $\beta<0$ can be determined via the von Karman–Howarth relation [Eq. (5)]: the viscous term can be neglected since shocks dominate the behavior of the velocity field. When there is no cutoff, the upper limit in the integrals in Eq. (5) is now $k_{\max} = 2\pi/\delta$, where δ is the shock thickness. For $-1<\beta<0$, it is easy to show that

$$S_3(r) \propto \mathrm{sgn}(r)|r|^{-\beta}. \tag{13}$$

When there is no cutoff, one finds for $\beta = -1$ that

$$S_3(r) \sim r \ln|r|, \tag{14}$$

a result used by Chekhlov and Yakhot [3]. We emphasize that when there is no cutoff in the noise there is no value of β for which we obtain strict linearity of $S_3(r)$.

CONCLUSIONS

In summary, when the Burgers equation is driven with a stochastic noise at length scales larger than a cutoff, shocks dominate the behavior of the system and multifractality [$S_q(r) \propto r^{\zeta_q}$ with $\zeta_q = 1$] occurs in the inertial range for all values of the exponent β that characterizes the form of the noise variance. The case of $\beta = 2$ was studied in Ref. [4]. We have studied the general case analytically for $q = 2$, 3, and 4; we have used knowledge of the scaling of the short-distance cutoff, identified with the shock thickness, with the viscosity to argue for the linear behavior when $q = 2$. The linearity of $S_3(r)$ follows directly from the noise term, while the linearity of S_4 derives from the form of the three-point function, for which we have an ansatz in wave-vector space. Our analytical work is corroborated by numerical simulations. It is possible to examine Eq. (7) and analyze the behavior of higher-order structure functions. One can plausibly argue that the linear behavior in S_{n-1} induces linear behavior in the next order structure function $S_n(r)$, as we demonstrated in the case of S_4. We contrasted this linear behavior of $S_2(r)$ and $S_3(r)$ with the corresponding behavior in the model in the absence of a noise cutoff: In our previous work [6] we had shown that multifractality occurred only for negative values of β and obtained $\zeta_2 = -2\beta/3$ numerically. Here we have obtained ζ_3 analytically. The values of ζ_2 and ζ_3 differ from the value of unity that occurs when there is a cutoff in the noise.

ACKNOWLEDGMENT

We thank the Ohio Supercomputer Center for providing time on the Cray-YMP which made this study possible.

[1] J. M. Burgers, *The Nonlinear Diffusion Equation: Asymptotic Solutions and Statistical Problems* (Reidel, Boston, 1977).
[2] A. Chekhlov and V. Yakhot, Phys. Rev. E **51**, R2739 (1995).
[3] A. Chekhlov and V. Yakhot, Phys. Rev. E **52**, 5681 (1995).
[4] J. P. Bouchaud, M. Mézard, and G. Parisi, Phys. Rev. E **52**, 3656 (1995).
[5] A. M. Polyakov, Phys. Rev. E **52**, 6183 (1995).
[6] F. Hayot and C. Jayaprakash, Phys. Rev. E **54**, 4681 (1996).
[7] For a superb discussion of various topics in turbulence, see U. Frisch, *Turbulence* (Cambridge, New York, 1995).
[8] P. G. Saffman, in *Topics in Nonlinear Physics*, edited by N. J. Zabusky (Springer-Verlag, New York, 1968).
[9] E. A. Novikov, Zh. Eksp. Teor. Fiz. **47**, 1919 (1964); [Sov. Phys. JETP **20**, 1290 (1965)]; M. D. Donsker, in *Analysis in Function Space*, edited by W. T. Martin and I. Segal (MIT Press, Cambridge, MA, 1964).
[10] E. Medina, T. Hwa, M. Kardar, and Y. C. Zhang, Phys. Rev. A **39**, 3053 (1989).